Quantum gravity appendix: The Graphs of Quantum Geometry

What does a graph have to do with geometry? Geometry implies measurements—things like the lengths of curves, the areas of surfaces, and the volumes of regions. A quantum geometry is called a geometry because it implies these kinds of measurements. Let us, for the sake of simplicity, consider quantum geometries in two dimensions. To do this, we limit the space the graph is embedded in to a two-dimensional surface. We also restrict the graph so that each vertex has three edges attached to it (see figure). Then we can reason as follows: For each graph, we associate a way to divide the two-dimensional space into triangles, like the surface of a geodesic dome.

Consider a single node, node A, with three edges attached to it. Let us place it in the center of a triangle (so that each of the three edges passes through one edge of the triangle. We say that the node A and its three edges are *dual* to the triangle T. Now let's choose one of the nodes that node A is connected to by an edge and call it node B. We associate a triangle with node B in the same way. Except that the edge connecting node A to node B is spoken for twice; it corresponds to the boundary between the two adjacent triangles. Let's consider those two lines to be identical.

We continue in this way to associate triangles with nodes and connect two triangles along a common line whenever the corresponding two nodes are joined by an edge of the graph. The result is a construction of a two-dimensional surface made up of triangles. We call this the *triangulation* of a surface.
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Each edge in the graph has an integer labeling it. That edge will cross a line where two triangles meet. We can consider that integer as denoting the length of the corresponding line in Planck units; thus if the edge is labeled 5, the line it crosses is five Planck lengths long. Therefore we know the length of each line of each triangle making up the triangulation of the surface. From this, we can calculate the area of each triangle and hence construct the geometry of the surface.

The same construction works in three dimensions with only a few changes. In this case, each node will have four edges attached. Instead of associating a triangle with a node, we will associate with each node a tetrahedron, which is a figure made by gluing four triangles together (see figure). We now take each node of the graph and put it in the center of a tetrahedron, with each of the four edges it's attached to going through one triangle face of the tetrahedron. Wherever two nodes are connected by an edge, we join the two tetrahedrons along the triangle corresponding to that edge.

When you're done following these instructions, you have a figure inscribed in a three-dimensional space made up of tetrahedrons glued together. This is called a triangulation of the three-dimensional space. Now each edge in the graph corresponds to a triangle. The area of that triangle is taken to be proportional to the number on the edge in units of the Planck length squared. In chapter 16, I mentioned that there are also labels on the nodes; these give the volume of the corresponding tetrahedron. So the graph and its labels give a system of triangles, each with an area, and a system of tetrahedrons, each with a volume. This can be called a quantum geometry.